Problem solving through paper folding

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The purpose of this article is to describe a couple of challenging mathematical problems that involve paper folding. These problem solving tasks can be used to foster geometric and algebraic thinking among students. The context of paper folding makes some of the abstract mathematical ideas involved relatively concrete. When implemented appropriately these activities have the potential to address many of the mathematical proficiencies, as delineated by Australian Curriculum and Assessment Reporting Authority (ACARA, 2014).

Mathematics is the study of order, relation and pattern. From its origins in counting and measuring it has evolved in highly sophisticated and elegant ways to become the language now used to describe many aspects of the world in the twenty-first century. Mathematics provides a framework for thinking and a means of communication that is powerful, logical, concise and precise. *Understanding*, *Fluency*, *Problem solving* and *Reasoning* are inherent in all aspects of this subject. Each of these proficiencies is essential, and all are mutually reinforcing. *Understanding* that a single mathematical process can be used in seemingly different situations, helps students to see the connections between different areas of study and encourages the transfer of learning. This is an important part of learning the art of mathematical *problem solving*. In performing such analyses, *reasoning* is required at each decision-making step and in drawing appropriate conclusions. Presenting the analysis in a logical and clear manner to explain the reasoning used is also an integral part of the learning process (ACARA, 2014).

Problem solving is an integral part of mathematics and mathematics education (ACARA, 2014; CCSSI, 2010; NCTM, 2000). Moreover, non-routine problems are a salient part any reformed mathematics classroom. However, non-routine problems are not easy to come by because once a non-routine problem becomes popular it has the potential to become relatively routine. Successful mathematics teachers are always thinking about and looking for rich non-routine mathematical problems and tasks. Deep reflection on mathematical ideas can help us come up with thought provoking non-routine mathematical problems.

The purpose of this paper is to illustrate how a simple rectangular sheet or strip of paper can be used to engage students in rich mathematical thinking in the context of geometry and algebra. Many crucial mathematical ideas can be visualised in the context of simple paper folding. Mathematical ideas like angle bisection, perpendicular bisector, congruence of shapes and segments, properties of right triangles, similar triangles, reflection, and rotation become more *tangible* and *vivid* in the context of paper folding. The activities discussed in this article can also be explored using dynamic geometry software. The following problems were formulated as the author was playing with a rather long rectangular strip of paper. The author believes that the problem is appropriate for secondary school students and first year university students taking a geometry class (ACARA, 2014; CCSSI, 2010). One of the strengths of the problems described in this article is that they can be modelled very easily with a long strip of paper. A rectangular sheet of printing paper cut into thirds lengthwise will play the role of manipulatives that will allow us to explore the problems discussed in this article. As a consequence, these problems are more hands-on in nature. The author would encourage the readers to try the problems first before reading the sketch of the solutions provided in this article.

The problem

Here is a description of the problem activity. As shown in Figure 1, let $ABCD$ represent a rather long rectangle strip of paper with $AB = m$, $AD = n$, where $m < n$. Points P and Q are on the lengths AD and BC, respectively such that $QC = a$, $PD = b$, where $a < b$. In Figure 2, the right-hand side trapezium portion of the rectangle strip is folded over the line segment PQ to form a crease along the line segment PQ. Note line segment $D'C'$ is the reflection of line segment DC along line segment PQ. We have to select m, n, a and b so that the entire side DC can be reflected along line segment PQ, and this reflection $D'C'$ falls below the side BC of the original rectangle $ABCD$. Point R is the intersection of BC and PD'. The problem is to find the exact perimeter and the exact area of the heptagon $APQC'D'RB$ created by folding rectangle $ABCD$.

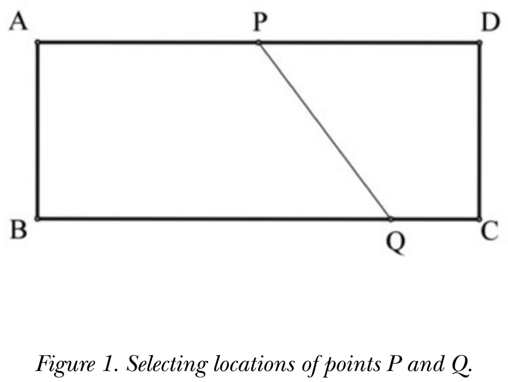


Figure 1. Selecting locations of points P and Q.

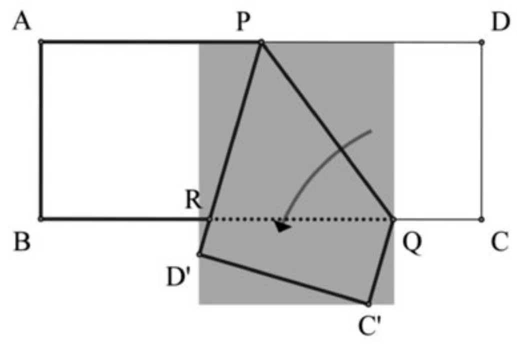


Figure 2. Heptagon created by folding along line segment PQ.

Here is a possible solution to the situation described above. To follow the solution please refer to Figure 2. Since $m\angle DPQ = m\angle PQR$, (because trapezium $PQC'D'$ is a reflection of trapezium $PQCD$ along the line segment PQ) and $m\angle DPQ = m\angle RPQ$ (because alternate interior angles are congruent when a transversals intersects two parallel lines), ΔPQR is an isosceles triangle with $m\angle PQR = m\angle RPQ$.

We can deduce the following by applying Pythagorean Theorem appropriately:

$$PQ = \sqrt{m^2 + (b-a)^2}$$

Suppose $PR = RQ = x$, then the following can be shown:

$$x^2 = m^2 + (x + a - b)^2$$
$$\Rightarrow x = \frac{m^2 + (b-a)^2}{2(b-a)}$$

Since $b > a$, x is positive.

Therefore, the perimeter of heptagon $APQC'D'RB$

$$\begin{aligned} &= 2(AB + AD) - PR - RQ + PQ \\ &= 2(m+n) - \frac{m^2 + (b-a)^2}{(b-a)} + \sqrt{m^2 + (b-a)^2} \end{aligned} \tag{1}$$

Note ΔPQR is an isosceles triangle with $m\angle PQR = m\angle RPQ$,

$$PQ = \sqrt{m^2 + (b-a)^2}$$

and
$$PR = RQ = \frac{m^2 + (b-a)^2}{2(b-a)}$$

Therefore, the height of ΔPQR from vertex R to side PQ

$$\begin{aligned} &= \sqrt{\left(\frac{m^2 + (b-a)^2}{2(b-a)}\right)^2 - \frac{m^2 + (b-a)^2}{4}} \\ &= \frac{m\sqrt{m^2 + (b-a)^2}}{2(b-a)} \end{aligned}$$

Therefore, the area of ΔPQR

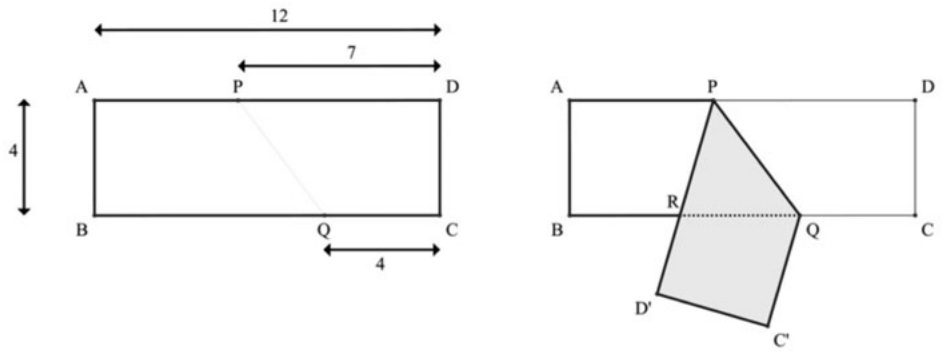
$$\begin{aligned} &= \frac{1}{2}\frac{m\sqrt{m^2 + (b-a)^2}}{2(b-a)}\sqrt{m^2 + (b-a)^2} \\ &= \frac{m^3 + m(b-a)^2}{4(b-a)} \end{aligned}$$

Therefore, the area of heptagon $APQC'D'RB$

$$= mn - \frac{m^3 + m(b-a)^2}{4(b-a)} \tag{2}$$

The general case

The author believes that it is important for teachers to come up with or see the general solution whenever possible because it will help them come up with more questions and the see the bigger picture. Figure 3 shows another concrete example of the general case that was discussed earlier (see Figures 1 and 2). The concrete version (as shown in Figure 3) may be more appropriate for the students. The pictures in Figure 3 are drawn to scale. The perimeter and the area of heptagon $APQC'D'RB$ in Figure 2 are $28\frac{2}{3}$ and $39\frac{2}{3}$, respectively.



Let $ABCD$ reprtesent a rather long rectangular strip of paper with $AB = 4$, $AD = 12$. Points P and Q are on the lengths AD and BC respectively such that $QC = 4$, $PD = 7$. The right-hand side trapezium portion of the rectangular strip is folded over the mine segment PQ to form a crease along the line segment PQ. Note line segment $D'C'$ is the reflection of line segment DC along line segment PQ. Point R is the inertsection of BC and PD'. Find the exact perimeter and the exact area of the heptagon $APQC'D'RB$ created by folding rectangle strip $ABCD$.

Figure 3. A concrete example of the general case.

The author has used both of these problems as a part of high school geometry courses and college-level geometry courses for pre-service teachers. These activities were well received by the students in both types of courses. The author also found the mathematical discussions students engaged in during these activities to be relatively rich. The author conjectures that this may be due to the fact that students have something that they can physically manipulate (that is, the rectangular strip of paper with the crease marks), so they find many of the related mathematical concepts to be more concrete and less abstract. The physical nature of the activity also helped the students communicate their mathematical ideas better.

References

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